

# Positive Solutions for Singular $m$ -Point Boundary Value Problems with Sign Changing Nonlinearities Depending on $x'$ \*

Ya Ma, Baoqiang Yan <sup>†</sup>

Department of Mathematics, Shandong Normal University, Jinan, 250014, P.R. China

## Abstract

Using the theory of fixed point theorem in cone, this paper presents the existence of positive solutions for the singular  $m$ -point boundary value problem

$$\begin{cases} x''(t) + a(t)f(t, x(t), x'(t)) = 0, 0 < t < 1, \\ x'(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \end{cases}$$

where  $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1, \alpha_i \in [0, 1], i = 1, 2, \dots, m-2$ , with  $0 < \sum_{i=1}^{m-2} \alpha_i < 1$  and  $f$  may change sign and may be singular at  $x = 0$  and  $x' = 0$ .

**Keywords:**  $m$ -point boundary value problem; Singularity; Positive solutions; Fixed point theorem

**Mathematics subject classification:** 34B15, 34B10

## 1. Introduction

The study of multi-point BVP (boundary value problem) for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [3-4]. Since then, many authors studied more general nonlinear multi-point BVP, for examples [2, 5-8], and references therein. In [7], Gupta, Ntouyas, and Tsamatos considered the existence of a  $C^1[0, 1]$  solution for the  $m$ -point boundary value problem

$$\begin{cases} x''(t) = f(t, x(t), x'(t)) + e(t), 0 < t < 1, \\ x'(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i), \end{cases}$$

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<sup>†</sup>Corresponding author: [yanbqcn@yahoo.com.cn](mailto:yanbqcn@yahoo.com.cn)

where  $\xi_i \in (0, 1)$ ,  $i = 1, 2, \dots, m-2$ ,  $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$ ,  $a_i \in R$ ,  $i = 1, 2, \dots, m-2$ , have the same sign,  $\sum_{i=1}^{m-2} a_i \neq 1$ ,  $e \in L^1[0, 1]$ ,  $f : [0, 1] \times R^2 \rightarrow R$  is a function satisfying Carathéodory's conditions and a growth condition of the form  $|f(t, u, v)| \leq p_1(t)|u| + q_1(t)|v| + r_1(t)$  with  $p_1, q_1, r_1 \in L^1[0, 1]$ . Recently, using Leray-Schauder continuation theorem, R.Ma and Donal O'Regan proved the existence of positive solutions of  $C^1[0, 1]$  solutions for the above BVP, where  $f : [0, 1] \times R^2 \rightarrow R$  satisfies the Carathéodory's conditions (see [8]).

Motivated by the works of [7,8], in this paper, we discuss the equation

$$\begin{cases} x''(t) + a(t)f(t, x(t), x'(t)) = 0, 0 < t < 1, \\ x'(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \end{cases} \quad (1.1)$$

where  $0 < \xi_i < 1$ ,  $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$ ,  $\alpha_i \in [0, 1)$  with  $0 < \sum_{i=1}^{m-2} \alpha_i < 1$  and  $f$  may change sign and may be singular at  $x = 0$  and  $x' = 0$ .

Our main features are as follows. Firstly, the nonlinearity  $af$  possesses singularity, that is,  $a(t)f(t, x, x')$  may be singular at  $t = 0$ ,  $t = 1$ ,  $x = 0$  and  $x' = 0$ ; also the degree of singularity in  $x$  and  $x'$  may be arbitrary (i. e., if  $f$  contains  $\frac{1}{x^\alpha}$  and  $\frac{1}{(-x')^\gamma}$ ,  $\alpha$  and  $\gamma$  may be big enough). Secondly,  $f$  is allowed to change sign. Finally, we discuss the maximal and minimal solutions for equations (1.1). Some ideas come from [11-12].

## 2. Preliminaries

Now we list the following conditions for convenience .

(H<sub>1</sub>)  $\beta, a, k \in C((0, 1), R_+)$ ,  $F \in C(R_+, R_+)$ ,  $G \in C(R_-, R_+)$ ,  $ak \in L[0, 1]$ ;

(H<sub>2</sub>)  $F$  is bounded on any interval  $[z, +\infty)$ ,  $z > 0$ ;

(H<sub>3</sub>)  $\int_{-\infty}^{-1} \frac{1}{G(y)} dy = +\infty$ ;

and the following conditions are satisfied

(P<sub>1</sub>)  $f \in C((0, 1) \times R_+ \times R_-, R)$ ;

(P<sub>2</sub>)  $0 < \sum_{i=1}^{m-2} \alpha_i < 1$ ,  $0 < \xi_i < 1$  and  $|f(t, x, y)| \leq k(t)F(x)G(y)$ ;

(P<sub>3</sub>) There exists  $\delta > 0$  such that  $f(t, x, y) \geq \beta(t)$ ,  $y \in (-\delta, 0)$ ;

where  $R_+ = (0, +\infty)$ ,  $R_- = (-\infty, 0)$ ,  $R = (-\infty, +\infty)$ .

**Lemma 2.1**<sup>[1]</sup> Let  $E$  be a Banach space,  $K$  a cone of  $E$ , and  $B_R = \{x \in E : \|x\| < R\}$ , where  $0 < r < R$ . Suppose that  $F: K \cap \overline{B_R} \setminus \overline{B_r} \rightarrow K$  is a completely continuous operator and the following conditions are satisfied

(1)  $\|F(x)\| \geq \|x\|$  for any  $x \in K$  with  $\|x\| = r$ .

(2) If  $x \neq \lambda F(x)$  for any  $x \in K$  with  $\|x\| = R$  and  $0 < \lambda < 1$ .

Then  $F$  has a fixed point in  $K_{R,r}$ .

Let  $C[0, 1] = \{x : [0, 1] \rightarrow R | x(t) \text{ is continuous on } [0, 1]\}$  with norm  $\|y\| = \max_{t \in [0, 1]} |y(t)|$ . Then  $C[0, 1]$  is a Banach space.

**Lemma 2.2** Let  $(H_1)$ -( $P_3$ ) hold. For each given natural number  $n > 0$ , there exists  $y_n \in C[0, 1]$  with  $y_n(t) \leq -\frac{1}{n}$  such that

$$y_n(t) = -\frac{1}{n} - \int_0^t a(s)f(s, (Ay_n)(s) + \frac{1}{n}, y_n(s))ds, \quad t \in [0, 1], \quad (2.1)$$

where

$$(Ay)(t) = \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i} \int_0^1 -y(\tau)d\tau - \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} -y(\tau)d\tau}{1 - \sum_{i=1}^{m-2} \alpha_i} - \int_0^t -y(\tau)d\tau, \quad t \in [0, 1].$$

**Proof.** For  $y \in P = \{y \in C[0, 1] : y(t) \leq 0, t \in [0, 1]\}$ , define a operator as follows

$$(T_n y)(t) = -\frac{1}{n} + \min\{0, -\int_0^t a(s)f(s, (Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds\}, \quad t \in [0, 1], \quad (2.2)$$

where  $n > 0$  is a natural number. For  $y \in P$ , we have

$$\begin{aligned} (Ay)(t) &= \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i} \int_0^1 -y(\tau)d\tau - \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} -y(\tau)d\tau}{1 - \sum_{i=1}^{m-2} \alpha_i} - \int_0^t -y(\tau)d\tau \\ &\geq \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i} \int_0^1 -y(\tau)d\tau - \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} -y(\tau)d\tau}{1 - \sum_{i=1}^{m-2} \alpha_i} - \int_0^1 -y(\tau)d\tau \\ &\geq \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i} \int_0^1 -y(\tau)d\tau - \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_{m-2}} -y(\tau)d\tau}{1 - \sum_{i=1}^{m-2} \alpha_i} \\ &\geq \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i} \int_{\xi_{m-2}}^1 -y(\tau)d\tau \\ &\geq 0, \quad t \in [0, 1]. \end{aligned}$$

Let

$$\begin{aligned} c(y(t)) &= -\int_0^t a(s)f(s, (Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds, \quad t \in [0, 1], \\ c(y_k(t)) &= -\int_0^t a(s)f(s, (Ay_k)(s) + \frac{1}{n}, \min\{y_k(s), -\frac{1}{n}\})ds, \quad t \in [0, 1]. \end{aligned}$$

By the equality  $\min\{c, 0\} = \frac{c - |c|}{2}$ , it is easy to know

$$(T_n y)(t) = -\frac{1}{n} + \frac{c(y(t)) - |c(y(t))|}{2}, \quad t \in [0, 1].$$

Let  $y_k, y \in P$  with  $\lim_{k \rightarrow +\infty} \|y_k - y\| = 0$ . Then, there exists a constant  $h > 0$ , such that  $\|y_k\| \leq h$  and  $\|y\| \leq h$ . Thus,  $|\min\{y_k(s), -\frac{1}{n}\} - \min\{y(s), -\frac{1}{n}\}| \rightarrow 0$ , uniformly for  $s \in [0, 1]$  as  $k \rightarrow +\infty$ . Therefore,  $|(Ay_k)(s) + \frac{1}{n} - ((Ay)(s) + \frac{1}{n})| \rightarrow 0$  for all  $s \in [0, 1]$  as  $k \rightarrow +\infty$ . ( $P_1$ ) implies that  $\{a(s)f(s, (Ay_k)(s) + \frac{1}{n}, \min\{y_k(s), -\frac{1}{n}\})\} \rightarrow \{a(s)f(s, (Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})\}$ , for  $s \in (0, 1)$  as  $k \rightarrow +\infty$ . By the Lebesgue dominated convergence

theorem (the dominating function  $a(s)k(s)F[\frac{1}{n}, +\infty)G[-h - \frac{1}{n}, -\frac{1}{n}]$ ), we have  $\|cy_k - cy\| \rightarrow 0$ , which yields that

$$\begin{aligned}\|T_n y_k - T_n y\| &= \left\| \frac{c(y_k) - c(y) - |c(y_k)| + |c(y)|}{2} \right\| \\ &\leq \left\| \frac{c(y_k) - c(y) + |c(y_k) - c(y)|}{2} \right\| \\ &\leq \|c(y_k) - c(y)\| \rightarrow 0, \text{ as } k \rightarrow +\infty.\end{aligned}$$

Consequently,  $T_n$  is a continuous operator.

Let  $C$  be a bounded set in  $P$ , i.e., there exists  $h_1 > 0$  such that  $\|y\| \leq h_1$ , for any  $y \in C$ . For any  $t_1, t_2 \in [0, 1], t_1 < t_2, y \in C$ ,

$$\begin{aligned}& |(T_n y)(t_2) - (T_n y)(t_1)| \\ &= \left| \frac{-\int_{t_1}^{t_2} a(s)f(s, (Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds}{2} \right. \\ &\quad \left. + \frac{|\int_0^{t_2} a(s)f(s, (Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds| - |\int_0^{t_1} a(s)f(s, (Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds|}{2} \right| \\ &\leq \left| \frac{-\int_{t_1}^{t_2} a(s)f(s, (Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds}{2} \right| \\ &\quad + \frac{|\int_{t_1}^{t_2} a(s)f(s, (Ay)(s), \min\{y(s), -\frac{1}{n}\})ds|}{2} \\ &\leq \left| \int_{t_1}^{t_2} a(s)k(s)ds \right| \sup F[\frac{1}{n}, +\infty) \sup G[-h_1 - \frac{1}{n}, -\frac{1}{n}].\end{aligned}$$

According to the absolute continuity of the Lebesgue integral, for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $|\int_{t_1}^{t_2} a(s)k(s)ds| < \epsilon, |t_2 - t_1| < \delta$ . Therefore,  $\{T_n y, y \in C\}$  is equicontinuous.

$$\begin{aligned}& |(T_n y)(t)| = \left| -\frac{1}{n} + \min\{0, -\int_0^t a(s)f(s, (Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds\} \right| \\ &\leq \left| \frac{1}{n} \right| + \left| \int_0^t a(s)f(s, (Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds \right| \\ &\leq 1 + \int_0^t a(s)|f(s, (Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})|ds \\ &\leq 1 + \int_0^1 a(s)k(s)ds \sup F[\frac{1}{n}, +\infty)G[-h - \frac{1}{n}, \frac{1}{n}], \quad t \in [0, 1].\end{aligned}$$

Therefore  $\{T_n y, y \in C\}$  is bounded.

Hence  $T_n$  is a completely continuous operator.

By (H<sub>3</sub>), choose a sufficiently large  $R_n > 1$  to fit  $\int_{-R_n}^{-1} \frac{dy}{G(y)} > \int_0^1 a(s)k(s)ds \sup F[\frac{1}{n}, +\infty)$ .

For  $n > \frac{1}{\delta}$ , we prove that

$$y(t) \neq \lambda(T_n y)(t) = \frac{-\lambda}{n} + \lambda \min\{0, -\int_0^t a(s)f(s, (Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds\}, \quad t \in [0, 1], \quad (2.3)$$

for any  $y \in P$  with  $\|y\| = R_n$  and  $0 < \lambda < 1$ .

In fact, if there exists  $y \in P$  with  $\|y\| = R_n$  and  $0 < \lambda < 1$  such that

$$y(t) = \frac{-\lambda}{n} + \lambda \min\{0, -\int_0^t a(s)f(s, (Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds\}, \quad t \in [0, 1]. \quad (2.4)$$

$y(0) = \frac{-\lambda}{n}$ . Since  $n > \frac{1}{\delta}$ , we have  $-\delta < y(0) < 0$ , which implies there exists  $\delta_0 > 0$  such that  $y(t) > -\delta, t \in (0, \delta_0)$ . (P<sub>3</sub>) implies

$$\int_0^t a(s)f(s, (Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds > 0, \quad t \in [0, 1].$$

Let  $t^* = \sup\{s \in [0, 1] \mid \int_0^t a(\tau)f(\tau, (Ay)(\tau) + \frac{1}{n}, \min\{y(\tau), -\frac{1}{n}\})d\tau > 0, 0 \leq t \leq s\}$ .

We show that  $t^* = 1$ . If  $t^* < 1$ , we have

$$\begin{cases} \int_0^t a(s)f(s, (Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds > 0, & t \in (0, t^*), \\ \int_0^t a(s)f(s, (Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds = 0, & t = t^*, \end{cases}$$

$$y(t) = \frac{-\lambda}{n} - \lambda \int_0^t a(s)f(s, (Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds, \quad t \in (0, t^*], \quad (2.5)$$

$$y(t^*) = \frac{-\lambda}{n} > -\delta. \quad (2.6)$$

(2.6) and (P<sub>3</sub>) imply there exists  $r > 0$  such that  $f(t, x, y) \geq \beta(t), t \in (t^* - r, t^*)$ . So

$$\begin{aligned} y(t^*) &= \frac{-\lambda}{n} - \lambda \int_0^{t^*} a(s)f(s, (Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds \\ &\leq \frac{-\lambda}{n} - \lambda \int_0^{t^*-r} a(s)f(s, (Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds - \lambda \int_{t^*-r}^{t^*} a(s)\beta(s)ds, \end{aligned}$$

$$\int_0^{t^*-r} a(s)f(s, (Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds + \int_{t^*-r}^{t^*} a(s)\beta(s)ds < 0,$$

which is a contradiction. Then,  $t^* = 1$ . Hence,

$$y(t) = \frac{-\lambda}{n} - \lambda \int_0^t a(s)f(s, (Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds, \quad t \in [0, 1]. \quad (2.7)$$

Since  $\|y\| = R_n > 1$  and  $y \in P$ , there exists a  $t_0 \in (0, 1)$  with  $y(t_0) = -R_n < -1$  and a  $t_1 \in (0, 1)$  such that  $y(t) < -1 < -\frac{1}{n}, t \in (t_0, t_1]$ , which together with (2.7) implies that

$$y(t) = \frac{-\lambda}{n} - \lambda \int_0^t a(s)f(s, (Ay)(s) + \frac{1}{n}, y(s))ds, \quad t \in (t_0, t_1]. \quad (2.8)$$

Differentiating (2.8) and using (H<sub>2</sub>), we obtain

$$-y'(t) = \lambda a(t)f(t, (Ay)(t) + \frac{1}{n}, y(t)) \leq a(t)F((Ay)(t) + \frac{1}{n})G(y(t)), \quad t \in (t_0, t_1].$$

And then

$$\frac{-y'(t)}{G(y(t))} \leq a(t)k(t) \sup F[(Ay)(t) + \frac{1}{n}, +\infty) \leq a(t)k(t) \sup F[\frac{1}{n}, +\infty), \quad t \in (t_0, t_1). \quad (2.9)$$

Integrating for (2.9) from  $t_0$  to  $t_1$ , we have

$$\int_{y(t_0)}^{y(t_1)} \frac{dy}{G(y)} \leq \int_{t_0}^{t_1} a(s)k(s)ds \sup F[\frac{1}{n}, +\infty), \quad t \in (t_0, t_1). \quad (2.10)$$

Then

$$\int_{-R_n}^{-1} \frac{dy}{G(y)} \leq \int_{-R_n}^{y(t_1)} \frac{dy}{G(y)} \leq \int_{t_0}^{t_1} a(s)k(s)ds \sup F[\frac{1}{n}, +\infty) \leq \int_0^1 a(s)k(s)ds \sup F[\frac{1}{n}, +\infty),$$

which contradicts

$$\int_{-R_n}^{-1} \frac{dy}{G(y)} > \int_0^1 a(s)k(s)ds \sup F[\frac{1}{n}, +\infty).$$

Hence (2.3) holds. Then put  $r = \frac{1}{n}$ , Lemma 2.1 leads to the desired result. This completes the proof.

**Lemma 2.3**<sup>[10]</sup> Let  $\{x_n(t)\}$  be an infinite sequence of bounded variation function on  $[a, b]$  and  $\{x_n(t_0)\}$  ( $t_0 \in [a, b]$ ) and  $\{V(x_n)\}$  be bounded ( $V(x)$  denotes the total variation of  $x$ ). Then there exists a subsequence  $\{x_{n_k}(t)\}$  of  $\{x_n(t)\}$ ,  $i \neq j, n_i \neq n_j$ , such that  $\{x_{n_k}(t)\}$  converges everywhere to some bounded variation function  $x(t)$  on  $[a, b]$ .

**Lemma 2.4**<sup>[9]</sup> (Zorn) If  $X$  is a partially ordered set in which every chain has an upper bound, then  $X$  has a maximal element.

### 3. Main results

**Theorem 3.1** Let  $(H_1)$ -( $P_3$ ) hold. Then the  $m$ -point boundary value problem (1.1) has at least one positive solution.

**Proof.** Put  $M_n = \min\{y_n(t) : t \in [0, \xi_{m-2}]\}$ ,  $(H_1)$  implies  $\gamma = \sup\{M_n\} < 0$ . In fact, if  $\gamma = 0$ , there exists  $n_k > N > 0$  such that  $M_{n_k} \rightarrow 0$  and  $-\delta < y_{n_k} < 0$ .  $(H_1)$  implies

$$\begin{aligned} y_{n_k}(t) &= -\frac{1}{n} - \int_0^t a(s)f(s, (Ay_{n_k})(s) + \frac{1}{n}, y_{n_k}(s))ds \\ &< -\frac{1}{n} - \int_0^t a(s)\beta(s)ds \\ &< -\int_0^t a(s)\beta(s)ds, \quad t \in [0, \xi_{m-2}]. \end{aligned}$$

Then  $y_{n_k}(\xi_{m-2}) < -\int_0^{\xi_{m-2}} a(s)\beta(s)ds$ , which contradicts to  $M_{n_k} \rightarrow 0$ .

Set  $\tau = \max\{\gamma, -\delta, -\int_0^{\xi_{m-2}} a(s)\beta(s)ds\}$ . In the remainder of the proof, assume  $n > -\frac{1}{\tau}$ .

1). First, we prove there exists a  $t_n \in (0, \xi_{m-2}]$  with  $y_n(t_n) = \tau$ . In fact, since  $y_n(0) = -\frac{1}{n} > \tau$ , there exists  $\delta_0 > 0$  such that  $y_n(t) > \tau, t \in (0, \delta_0)$ . Let  $t_n = \sup\{t | s \in$

$[0, t], y_n(s) > \tau\}$ . Then  $y_n(t_n) = \tau$ . If  $t_n > \xi_{m-2}$ , we have  $y_n(t) > \tau > -\delta, t \in [0, \xi_{m-2}]$ .  $(H_1)$  shows that

$$\begin{aligned} y_n(t) &= -\frac{1}{n} - \int_0^t a(s)f(s, (Ay_n)(s) + \frac{1}{n}, y_n(s))ds \\ &\leq -\frac{1}{n} - \int_0^t a(s)\beta(s)ds \\ &\leq -\int_0^t a(s)\beta(s)ds, \quad t \in [0, \xi_{m-2}]. \end{aligned}$$

Then  $\tau < y_n(\xi_{m-2}) \leq -\int_0^{\xi_{m-2}} a(s)\beta(s)ds < \tau$ , which is a contradiction.

Second, we prove

$$y_n(t) \leq \tau, \quad t \in [t_n, 1]. \quad (3.1)$$

In fact, if there exists a  $t \in (t_n, 1]$  such that  $y_n(t) > \tau$ , and we choose  $t', t'' \in [t_n, 1], t' < t''$  to fit  $y_n(t') = \tau, \tau < y_n(t) < -\frac{1}{n}, t \in (t', t'']$ , from (2.1)

$$0 < \int_{t'}^{t''} a(s)f(s, (Ay_n)(s) + \frac{1}{n}, y_n(s))ds = y_n(t') - y_n(t'') < 0.$$

This contradiction implies that (3.1) holds. Then

$$\begin{cases} y_n(t) \leq -\int_0^t a(s)\beta(s)ds, & t \in [0, t_n], \\ y_n(t) \leq \tau, & t \in [t_n, 1]. \end{cases}$$

Let  $W(t) = \max\{-\int_0^t a(s)\beta(s)ds, \tau\}, t \in (0, 1)$ . Obviously,  $W(t)$  is bounded on  $[\frac{1}{3k}, 1 - \frac{1}{3k}]$  and  $y_n(t) \leq W(t), t \in [0, 1]$ .

2).  $\{y_n(t)\}$  is equicontinuous on  $[\frac{1}{3k}, 1 - \frac{1}{3k}]$  ( $k \geq 1$  is a natural number) and uniformly bounded on  $[0, 1]$ .

Notice that

$$\begin{aligned} (Ay_n)(t) + \frac{1}{n} &= \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i} \int_0^1 -y_n(\tau)d\tau - \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} -y_n(\tau)d\tau}{1 - \sum_{i=1}^{m-2} \alpha_i} - \int_0^t -y_n(\tau)d\tau + \frac{1}{n} \\ &> \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i} \int_{\xi}^1 -y_n(\tau)d\tau \geq \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i} (-\tau)(1 - \xi) = \Theta, t \in [0, 1]. \end{aligned}$$

We know from (2.9)

$$\int_{y_n(t)}^{-\frac{1}{n}} \frac{dy_n}{G(y_n)} \leq \int_0^t a(s)k(s)ds \sup F[\Theta, +\infty), t \in [0, 1]. \quad (3.2)$$

Now  $(H_3)$  and (3.2) show that  $\omega(t) = \inf\{y_n(t)\} > -\infty$  is bounded on  $[0, 1]$ . On the other hand, it follows from (2.1) and (3.1) that

$$|y'_n(t)| \leq k(t)a(t) \sup F[\Theta, +\infty) \sup G[\omega_k, \max\{\tau, W(\frac{1}{3k})\}], \quad (n \geq k), \quad (3.3)$$

where  $\omega_k = \inf\{\omega(t), t \in [\frac{1}{3k}, 1 - \frac{1}{3k}]\}$ . Thus (3.3) and the absolute continuity of Lebesgue integral show that  $\{y_n(t)\}$  is equicontinuous on  $[\frac{1}{3k}, 1 - \frac{1}{3k}]$ . Now the Arzela-Ascoli theorem guarantees that there exists a subsequence of  $\{y_n^{(k)}(t)\}$ , which converges uniformly on  $[\frac{1}{3k}, 1 - \frac{1}{3k}]$ . When  $k = 1$ , there exists a subsequence  $\{y_n^{(1)}(t)\}$  of  $\{y_n(t)\}$ , which converges uniformly on  $[\frac{1}{3}, \frac{2}{3}]$ . When  $k = 2$ , there exists a subsequence  $\{y_n^{(2)}(t)\}$  of  $\{y_n^{(1)}(t)\}$ , which converges uniformly on  $[\frac{1}{6}, \frac{5}{6}]$ . In general, there exists a subsequence  $\{y_n^{(k+1)}(t)\}$  of  $\{y_n^{(k)}(t)\}$ , which converges uniformly on  $[\frac{1}{3(k+1)}, 1 - \frac{1}{3(k+1)}]$ . Then the diagonal sequence  $\{y_k^{(k)}(t)\}$  converges pointwise in  $(0, 1)$  and it is easy to verify that  $\{y_k^{(k)}(t)\}$  converges uniformly on any interval  $[c, d] \subseteq (0, 1)$ . Without loss of generality, let  $\{y_k^{(k)}(t)\}$  be itself of  $\{y_n(t)\}$  in the rest. Put  $y(t) = \lim_{n \rightarrow \infty} y_n(t), t \in (0, 1)$ . Then  $y(t)$  is continuous on  $(0, 1)$  and since  $y_n(t) \leq W(t) < 0$ , we have  $y(t) \leq 0, t \in (0, 1)$ .

3) Now (3.2) shows

$$\sup\{\max\{-y_n(t), t \in [0, 1]\}\} < +\infty.$$

We have

$$\lim_{t \rightarrow 0^+} \sup\{\int_0^t -y_n(s)ds\} = 0, \quad \lim_{t \rightarrow 1^-} \sup\{\int_t^1 -y_n(s)ds\} = 0, \quad t \in [0, 1], \quad (3.4)$$

and

$$\begin{aligned} (Ay_n)(t) &= \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i} \int_0^1 -y_n(\tau)d\tau - \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} -y_n(\tau)d\tau}{1 - \sum_{i=1}^{m-2} \alpha_i} - \int_0^t -y_n(\tau)d\tau \\ &< \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i} \int_0^1 -y_n(\tau)d\tau \\ &< +\infty, \quad t \in [0, 1]. \end{aligned} \quad (3.5)$$

Since (3.4) and (3.5) hold, the Fatou theorem of the Lebesgue integral implies  $(Ay)(t) < +\infty$ , for any fixed  $t \in (0, 1)$ .

4)  $y(t)$  satisfies the following equation

$$y(t) = - \int_0^t a(s)f(s, (Ay)(s), y(s))ds, \quad t \in (0, 1). \quad (3.6)$$

Since  $y_n(t)$  converges uniformly on  $[a, b] \subset (0, 1)$ , (3.4) implies that  $(Ay_n)(s)$  converges to  $(Ay)(s)$  for any  $s \in (0, 1)$ . For fixed  $t \in (0, 1)$  and any  $d, 0 < d < t$ , we have

$$y_n(t) - y_n(d) = - \int_d^t a(s)f(s, (Ay_n)(s) + \frac{1}{n}, y_n(s))ds. \quad (3.7)$$

for all  $n > k$ . Since  $y_n(s) \leq \max\{\tau, W(d)\}$ ,  $(Ay_n)(s) + \frac{1}{n} \geq \Theta$ ,  $s \in [d, t]$ ,  $\{(Ay_n)(s)\}$  and  $\{y_n(s)\}$  are bounded and equicontinuous on  $[d, t]$

$$y(t) - y(d) = - \int_d^t a(s)f(s, (Ay)(s), y(s))ds. \quad (3.8)$$



Putting  $t = d$  in (3.2), we have

$$\int_{y_n(d)}^{-\frac{1}{n}} \frac{dy_n}{G(y_n)} \leq \int_0^d a(s)k(s)ds \sup F[\Theta, +\infty). \quad (3.9)$$

Letting  $n \rightarrow \infty$  and  $d \rightarrow 0^+$ , we obtain

$$y(0^+) = \lim_{d \rightarrow 0^+} y(d) = 0.$$

Letting  $d \rightarrow 0^+$  in (3.8), we have

$$y(t) = - \int_0^t a(s)f(s, (Ay)(s), y(s))ds, \quad t \in (0, 1), \quad (3.10)$$

and

$$(Ay)(1) = \sum_{i=1}^{m-2} \alpha_i(Ay)(\xi_i).$$

Hence  $x(t) = (Ay)(t)$  is a positive solution of (1.1).  $\square$

**Theorem 3.2** Suppose that  $(H_1)$ -( $P_3$ ) hold. Then the set of positive solutions of (1.1) is compact in  $C^1[0, 1]$ .

**Proof** Let  $M = \{y \in C[0, 1]: (Ay)(t) \text{ is a positive solution of equation (1.1)}\}$ . We show that

- (1)  $M$  is not empty;
- (2)  $M$  is relatively compact (bounded, equicontinuous);
- (3)  $M$  is closed.

Obviously, Theorem 3.1 implies  $M$  is not empty.

First, we show that  $M \subset C[0, 1]$  is relatively compact. For any  $y \in M$ , differentiating (3.10) and using  $(H_2)$ , we obtain

$$\begin{aligned} -y'(t) &= a(t)f(t, (Ay)(t), y(t)) \\ &\leq a(t)|f(t, (Ay)(t), y(t))| \\ &\leq a(t)k(t)F[\Theta, +\infty)G(y(t)), \quad t \in (0, 1), \\ \frac{-y'(t)}{G(y(t))} &\leq a(t)k(t) \sup F[(Ay)(t), +\infty) \\ &\leq a(t)k(t) \sup F[\Theta, +\infty), \quad t \in [0, 1]. \end{aligned} \quad (3.11)$$

Integrating for (3.11) from 0 to  $t$ , we have

$$\int_{y(t)}^0 \frac{dy}{G(y)} \leq \int_0^1 a(s)k(s)ds \sup F[\Theta, +\infty), \quad t \in [0, 1]. \quad (3.12)$$

Now  $(H_3)$  and (3.12) show that for any  $y \in M$ , there exists  $K > 0$  such that  $|y(t)| < K, \forall t \in [0, 1]$ . Then  $M$  is bounded.

For any  $y \in M$ , we obtain from (3.11)

$$\begin{aligned} -y'(t) &= a(t)f(t, (Ay)(t), y(t)) \\ &\leq a(t)|f(t, (Ay)(t), y(t))| \\ &\leq a(t)k(t)F[\Theta, +\infty)G(y(t)), \quad t \in (0, 1), \end{aligned}$$

and

$$\begin{aligned} y'(t) &= -a(t)f(t, (Ay)(t), y(t)) \\ &\leq a(t)|f(t, (Ay)(t), y(t))| \\ &\leq a(t)k(t)F[\Theta, +\infty)G(y(t)), \quad t \in (0, 1), \end{aligned}$$

which yields

$$\frac{-y'(t)}{G(y(t)) + 1} \leq a(t)k(t) \sup F[\Theta, +\infty), \quad t \in (0, 1), \quad (3.13)$$

and

$$\frac{y'(t)}{G(y(t)) + 1} \leq a(t)k(t) \sup F[\Theta, +\infty), \quad t \in (0, 1). \quad (3.14)$$

Notice that the rights are always positive in (3.13) and (3.14). Let  $I(y(t)) = \int_0^{y(t)} \frac{dy}{G(y) + 1}$ . For any  $t_1, t_2 \in [0, 1]$ , integrating for (3.13) and (3.14) from  $t_1$  to  $t_2$ , we obtain

$$|I(y(t_1)) - I(y(t_2))| \leq \int_{t_1}^{t_2} a(t)k(t)F[\Theta, +\infty)dt. \quad (3.15)$$

Since  $I^{-1}$  is uniformly continuous on  $[I(-K), 0]$ , for any  $\bar{\epsilon} > 0$ , there is a  $\epsilon' > 0$  such that

$$|I^{-1}(s_1) - I^{-1}(s_2)| < \bar{\epsilon}, \quad \forall |s_1 - s_2| < \epsilon', \quad s_1, s_2 \in [I(-K), 0]. \quad (3.16)$$

And (3.15) guarantees that for  $\epsilon' > 0$ , there is a  $\delta' > 0$  such that

$$|I(y(t_1)) - I(y(t_2))| < \epsilon', \quad \forall |t_1 - t_2| < \delta', \quad t_1, t_2 \in [0, 1]. \quad (3.17)$$

Now (3.16) and (3.17) yield that

$$|y(t_1) - y(t_2)| = |I^{-1}(I(y(t_1))) - I^{-1}(I(y(t_2)))| < \bar{\epsilon}, \quad t_1, t_2 \in [0, 1], \quad (3.18)$$

which means that  $M$  is equicontinuous. So  $M$  is relatively compact.

Second, we show that  $M$  is closed. Suppose that  $\{y_n\} \subseteq M$  and  $\lim_{n \rightarrow +\infty} \max_{t \in [0, 1]} |y_n(t) - y_0(t)| = 0$ . Obviously  $y_0 \in C[0, 1]$  and  $\lim_{n \rightarrow +\infty} (Ay_n)(t) = (Ay_0)(t)$ ,  $t \in [0, 1]$ . Moreover,

$$\begin{aligned} (Ay_n)(t) &= \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i} \int_0^1 -y_n(\tau) d\tau - \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} -y_n(\tau) d\tau}{1 - \sum_{i=1}^{m-2} \alpha_i} - \int_0^t -y_n(\tau) d\tau \\ &< \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i} \int_0^1 -y_n(\tau) d\tau \\ &< \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i}, \quad t \in [0, 1]. \end{aligned} \quad (3.19)$$

For  $y_n \in M$ , from (3.10) we obtain

$$y_n(t) = - \int_0^t a(s)f(s, (Ay_n)(s), y_n(s))ds, \quad t \in (0, 1). \quad (3.20)$$

For fixed  $t \in (0, 1)$ , there exists  $0 < d < t$  such that

$$y_n(t) - y_n(d) = - \int_d^t a(s)f(s, (Ay_n)(s), y_n(s))ds. \quad (3.21)$$

Since  $y_n(s) \leq \max\{\tau, W(d)\}$ ,  $(Ay_n)(s) \geq \Theta$ ,  $s \in [d, t]$ , the Lebesgue Dominated Convergence Theorem yields that

$$y_0(t) - y_0(d) = - \int_d^t a(s)f(s, (Ay_0)(s), y_0(s))ds, \quad t \in (0, 1). \quad (3.22)$$

From (3.10), we have

$$\begin{aligned} -y'_n(t) &= a(t)f(t, (Ay_n)(s), y_n(s)) \\ &\leq a(t)k(t)F[\Theta, +\infty)G(y_n(t)), \quad t \in (0, 1), \end{aligned}$$

which yields

$$\frac{-y'_n(t)}{G(y_n(t))} \leq a(t)k(t) \sup F[\Theta, +\infty), \quad t \in (0, 1).$$

Integrating from 0 to  $d$

$$\int_{y_n(d)}^0 \frac{dy_n}{G(y_n)} \leq \int_0^d a(s)k(s)ds \sup F[\Theta, +\infty). \quad (3.23)$$

Letting  $n \rightarrow \infty$  and  $d \rightarrow 0^+$ , we obtain

$$y_0(0^+) = \lim_{d \rightarrow 0^+} y_0(d) = 0.$$

Letting  $d \rightarrow 0^+$  in (3.22), we have

$$y_0(t) = - \int_0^t a(s)f(s, (Ay_0)(s), y_0(s))ds, \quad t \in (0, 1), \quad (3.24)$$

and

$$(Ay_0)(1) = \sum_{i=1}^{m-2} \alpha_i (Ay_0)(\xi_i).$$

Then  $x_0(t) = (Ay_0)(t)$  is a positive solution of (1.1). So  $y_0 \in M$  and  $M$  is a closed set.

Hence  $\{Ay, y \subseteq M\} \in C^1[0, 1]$  is compact.

**Theorem 3.3** Suppose  $(H_1)$ -( $P_3$ ) hold. Then (1.1) has a minimal positive solution and a maximal positive solution in  $C^1[0, 1]$ .

**Proof.** Let  $\Omega = \{x(t) : x(t) \text{ is a } C^1[0, 1] \text{ positive solution of (1.1)}\}$ . Theorem 3.1 implies that  $\Omega$  is nonempty. Define a partially ordered  $\leq$  in  $\Omega$  :  $x \leq y$  iff  $x(t) \leq y(t)$  for any  $t \in [0, 1]$ . We prove only that any chain in  $(\Omega, \leq)$  has a lower bound in  $\Omega$ . The rest is obtained from Zorn's lemma. Let  $\{x_\alpha(t)\}$  be a chain in  $(\Omega, \leq)$ . Since  $C[0, 1]$  is a separable Banach space, there exists countable set at most  $\{x_n(t)\}$ , which is dense in  $\{x_\alpha(t)\}$ . Without loss of generality, we may assume that  $\{x_n(t)\} \subseteq \{x_\alpha(t)\}$ . Put  $z_n(t) = \min\{x_1(t), x_2(t), \dots, x_n(t)\}$ . Since  $\{x_\alpha(t)\}$  is a chain,  $z_n(t) \in \Omega$  for any  $n$  (in fact,  $z_n(t)$  equals one of  $x_n(t)$ ) and  $z_{n+1}(t) \leq z_n(t)$  for any  $n$ . Put  $z(t) = \lim_{m \rightarrow +\infty} z_m(t)$ . We prove that  $z(t) \in \Omega$ .

By Lemma 2.2, there exists  $y_n(t)$  (e.g.,  $y_n(t)$  may be  $z'_n(t)$ ), which is a solution of

$$(Ty)(t) = - \int_0^t a(s)f(s, (Ay)(s), y(s))ds \quad t \in [0, 1],$$

such that

$$z_n(t) = \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i} \int_0^1 -y_n(\tau) d\tau - \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} -y_n(\tau) d\tau}{1 - \sum_{i=1}^{m-2} \alpha_i} - \int_0^t -y_n(\tau) d\tau.$$

(3.2) imply that  $\{\|y_n\|\}$  is bounded. From Lemma 2.3, there exists a subsequence  $\{y_{n_k}(t)\}$  of  $\{y_n(t)\}$ ,  $i \neq j, n_i \neq n_j$ , which converges everywhere on  $[0, 1]$ . Without loss of generality, let  $\{y_{n_k}(t)\}$  be itself of  $\{y_n(t)\}$ . Put  $y_0(t) = \lim_{m \rightarrow +\infty} y_n(t), t \in [0, 1]$ . Use  $y_n(t), y_0(t)$ , and 0 in place of  $y_n(t), y(t)$ , and  $1/n$  in Theorem 3.1, respectively. A similar argument to show Theorem 3.1 yields that  $y_0(t)$  is a solution of

$$y(t) = - \int_0^t a(s) f(s, (Ay)(s), y_n(s)) ds, \quad t \in [0, 1].$$

The boundedness of  $\{\|y_n\|\}$  leads to

$$\begin{aligned} z(t) &= \lim_{m \rightarrow +\infty} z_n(t) \\ &= \lim_{m \rightarrow +\infty} \left[ \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i} \int_0^1 -y_n(\tau) d\tau - \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} -y_n(\tau) d\tau}{1 - \sum_{i=1}^{m-2} \alpha_i} - \int_0^t -y_n(\tau) d\tau \right] \\ &= \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i} \int_0^1 -y_0(\tau) d\tau - \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} -y_0(\tau) d\tau}{1 - \sum_{i=1}^{m-2} \alpha_i} - \int_0^t -y_0(\tau) d\tau. \end{aligned}$$

Hence  $z \in \Omega$ . By Lemma 2.2, for any  $x \in \{x_\alpha\}$ , there exists  $\{x_{n_k}\} \subseteq \{x_n\}$  such that  $\|x_{n_k} - x\| \rightarrow 0$ . Notice that  $x_{n_k}(t) \geq z_{n_k}(t) \geq z(t), t \in [0, 1]$ . Letting  $k \rightarrow +\infty$ , we have  $x(t) \geq z(t), t \in [0, 1]$ ; i.e.,  $\{x_\alpha\}$  has lower boundedness in  $\Omega$ . Zorn's lemma shows that (1.1) has a minimal  $C^1[0, 1]$  positive solution. By a similar proof, we can get the a maximal  $C^1[0, 1]$  positive solution. The proof is complete.

**Theorem 3.4** Suppose that  $(H_1)$ -( $P_3$ ) hold,  $f(t, x, z)$  is decreasing in  $x$  for all  $(t, z) \in [0, 1] \times R_-$ ,  $a(0)f(0, x, z) \neq 0$  and  $\lim_{t \rightarrow 0} f(t, x, y) \neq +\infty$ . Then (1.1) has an unique positive solution in  $C^1[0, 1]$ .

**Proof.** Assume that  $x_1$  and  $x_2$  are two positive different solutions to (1.1), i.e., there exists  $t_0 \in (0, 1]$  such that  $x_1(t_0) \neq x_2(t_0)$ . Without loss of generality, assume that  $x_1(t_0) > x_2(t_0)$ . Let  $\varphi(t) = x_1(t) - x_2(t)$  for all  $t \in [0, 1]$ . Obviously,  $\varphi \in C[0, 1] \cap C^1(0, 1]$  with  $\varphi(t_0) > 0$ .

Let  $t_* = \inf\{0 < t < t_0 | \varphi(s) > 0 \text{ for all } s \in [t, t_0]\}$  and  $t^* = \sup\{t_0 < t < 1 | \varphi(s) > 0 \text{ for all } s \in [t, t^*]\}$ . It is easy to see that  $\varphi(t) > 0$  for all  $t \in (t_*, t^*)$  and  $\varphi$  has maximum in  $[t_*, t^*]$ . Let  $t'$  satisfying that  $\varphi(t') = \max_{t \in [t_*, t^*]} \varphi(t)$ . There are three cases: (1)  $t' \in (t_*, t^*)$ ; (2)  $t' = t^* = 1$ ; (3)  $t' = 0$ .

(1)  $t' \in (t_*, t^*)$ . It is easy to see that  $\varphi''(t') \leq 0$  and  $\varphi'(t') = 0$ . Then  $\varphi''(t') = x_1''(t') - x_2''(t')$

$$= -a(t')f(t', x_1(t'), x_1'(t')) + a(t')f(t', x_2(t'), x_2'(t')) > 0,$$

a contradiction.

(2)  $t' = t^* = 1$ . Since  $t' = t^* = 1$ , we have  $\sum_{i=1}^{m-2} \alpha_i \max\{\varphi(\xi_i)\} > \sum_{i=1}^{m-2} \alpha_i \varphi(\xi_i) = \varphi(1)$ , a

contradiction to  $0 < \sum_{i=1}^{m-2} \alpha_i < 1$ .

(3)  $t' = 0$ . Since  $t' = 0$  and  $x_1$  and  $x_2$  are solutions, the proof of lemma 2.2 implies that there exist  $x_{n,1}$  and  $x_{n,2}$  such that

$$\|x_{n,1} - x_1\| < \frac{\varphi(0)}{2}, \quad \|x_{n,2} - x_2\| < \frac{\varphi(0)}{2}$$

where

$$x_{n,1}(t) = \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i} \int_0^1 -y_{n,1}(\tau) d\tau - \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} -y_{n,1}(\tau) d\tau}{1 - \sum_{i=1}^{m-2} \alpha_i} - \int_0^t -y_{n,1}(\tau) d\tau, \quad t \in [0, 1],$$

$$x_{n,2}(t) = \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i} \int_0^1 -y_{n,2}(\tau) d\tau - \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} -y_{n,2}(\tau) d\tau}{1 - \sum_{i=1}^{m-2} \alpha_i} - \int_0^t -y_{n,2}(\tau) d\tau, \quad t \in [0, 1],$$

and

$$y_{n,1}(t) = -\frac{1}{n} - \int_0^t a(s) f(s, x_{n,1}(s) + \frac{1}{n}, y_{n,1}(s)) ds, \quad t \in [0, 1],$$

$$y_{n,2}(t) = -\frac{1}{n} - \int_0^t a(s) f(s, x_{n,2}(s) + \frac{1}{n}, y_{n,2}(s)) ds, \quad t \in [0, 1],$$

$$y_{n,1}(t) \leq -\frac{1}{n}, \quad y_{n,2}(t) \leq -\frac{1}{n} \text{ for all } t \in [0, 1].$$

By a similar proof with above, there exists  $t_1 \in (0, 1]$  such that  $x_{n,1}(t_1) \neq x_{n,2}(t_1)$ . Without loss of generality, assume that  $x_{n,1}(t_1) > x_{n,2}(t_1)$ . Let  $\varphi_n(t) = x_{n,1}(t) - x_{n,2}(t)$  for all  $t \in [0, 1]$ . Obviously,  $\varphi_n \in C[0, 1] \cap C^1(0, 1]$  with  $\varphi_n(t_1) > 0$ . Let  $t_* = \inf\{0 < t < t_1 | \varphi_n(s) > 0 \text{ for all } s \in [t, t_1]\}$  and  $t^* = \sup\{t_1 < t < 1 | \varphi_n(s) > 0 \text{ for all } s \in [t_1, t]\}$ . It is easy to see that  $\varphi_n(t) > 0$  for all  $t \in (t_1^*, t^{1*})$  and  $\varphi_n$  has maximum in  $[t_1^*, t^{1*}]$ . Let  $t''$  satisfying that  $\varphi(t'') = \max_{t \in [t_1^*, t^{1*}]} \varphi(t)$ . There are three cases: 1)  $t'' \in (t_1^*, t^{1*})$ ; 2)  $t'' = t^* = 1$ ; 3)  $t'' = 0$ .

The proof of 1) and 2) are similar with (1) and (2).

3)  $t'' = 0$ . We have  $\varphi_n(t) < \varphi_n(0)$ ,  $t \in (0, 1]$ ,  $\varphi'_n(0) = 0$ ,  $\varphi'_n(t_\xi) < 0$ ,  $t_\xi \in (0, 1)$ . Then

$$\lim_{t_\xi \rightarrow 0+} \varphi''_n(t) = \lim_{t_\xi \rightarrow 0+} \frac{\varphi'_n(t_\xi) - \varphi'_n(0)}{t_\xi - 0} \leq 0.$$

On the other hand, since  $\varphi''_n(0) = x''_{n,1}(0) - x''_{n,2}(0)$

$$= -a(0)f(0, x_{n,1}(0) + \frac{1}{n}, x'_{n,1}(0)) + a(0)f(0, x_{n,2}(0) + \frac{1}{n}, x'_{n,2}(0)) > 0,$$

a contradiction. Then (1.1) has at most one solution. The proof is complete.

**Example 3.1.** In (1.1), let  $f(t, x, y) = k(t)[1 + x^{-\gamma} + (-y)^{-\sigma} - (-y) \ln(-y)]$ ,  $a(t) = t^{-\frac{1}{3}}$ , and

$$k(t) = t^{-\frac{1}{2}}, \quad 0 < t < 1,$$

where  $\gamma > 0, \sigma < -2$ , and let  $F(x) = 1 + x^{-\gamma}, G(y) = 1 + (-y)^{-\sigma} - (-y) \ln(-y)$ . Then

$$f(t, x, y) \leq k(t)F(x)G(y), \quad \delta = 1, \quad \beta(t) = k(t),$$

and

$$\int_{-\infty}^{-1} \frac{dy}{G(y)} = +\infty.$$

By Theorem 3.1, (1.1) at least has a positive solution and Corollary 3.1 implies the set of solutions is compact.

## References

- [1] K.Deimling, Nonlinear Functional Analysis, *Springer -Verlag, Berlin*, 1985.
- [2] Y.Guo, W.Ge, Positive solutions for three-point boundary value problems with dependence on the first order derivative, *J. Math. Anal. Appl.*, **290**(2004), pp291-301.
- [3] V.A.Il'in, E.I.Moiseev, Nonlocal boundary value problem of the second kind for a Sturm-Liouville operator, *Differential Equation* **23 (8)**(1987), pp979-987.
- [4] V.A.Il'in, E.I.Moiseev, Nonlocal value problem of the first kind for a Sturm-Liouville operator in its differential and finite difference aspects, *Differential Equations* **23 (7)** (1987), pp803-810.
- [5] B.Liu, Positive solutions of a nonlinear three-point boundary value problem, *Applied Mathematics and Computation* **132** (2002), pp11-28.
- [6] R.Ma, Positive solutions for a nonlinear three-point boundary value problem, *Electron. J.Differential Equations* **34** (1999), pp1-8.
- [7] C.P. Gupta, S.K. Ntouyas, P.Ch. Tsamatos, Solvability of an  $m$ -point boundary value problem for second order ordinary differential equations, *J. Math. Anal. Appl.* **189** (1995) pp575-584.
- [8] R.Ma, Donal O'Regan, Solvability of singular second order  $m$ -point boundary value problems *Journal of Mathematical Analysis and Applications* **301**(2005), pp124-134.
- [9] Mario Petrich and Norman Reilly, Completely regular semigroups, Canadian Mathematical Society series of monographs, *Springer New York* **64** (2002).
- [10] D. Xia et al., Foundation of Real Variable Function and Applied Functional Analysis, *Shanghai Science and Technology Press, Shanghai*, (1987) (in Chinese).
- [11] G.Yang, Minimal Positive Solutions to Some Singular Second-Order Differential Equations, *Computers J. Math. Anal. Appl.* **266** (2002), pp479-491.
- [12] G.Yang, Positive solutions of some second order nonlinear singular differential equations, *Computers Math.Appl.* **45** (2003), pp604-615.
- [13] G.Yang, Second order singular boundary value problems with sign-changing nonlinearities on infinity intervals. *Nonlinear Analysis Forum.* **9 (2)** (2004), pp169-174.

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